A solution to multi-dimensional diffusion

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Abstract: Diffusions in multi-dimensions are considered. In all cases, particle densities decrease quasi-exponentially with the same decay constant.

Prologue

It is generally known that an infinitely long meteor trail decays exponentially [McKinley, 1961]. The question we like to answer is that if a pile of particles, e.g. plasma from meteoric ablation, is deposited in a spherical region, how does it diffuse. If they are confined in a tube, how does it diffuse?

Theory

The governing equation for diffusion comes from the continuity equation, i.e.,

\[
\frac{\partial n}{\partial t} = -\text{div}(nV)
\]  

(1)

where, \(n\) is the particle density, \(V\) is the velocity vector, and \(t\) is time variable. In the case of diffusion, we have

\[
V = \frac{D}{n} \text{grad}(n)
\]  

(2)

where \(D\) is the diffusion coefficient. Substituting (2) into (1), we have the diffusion equation as

\[
\frac{\partial n}{\partial t} = D \nabla^2 n = D \left( \frac{\partial^2 n}{\partial x_1^2} + \frac{\partial^2 n}{\partial x_2^2} + \ldots + \frac{\partial^2 n}{\partial x_n^2} \right)
\]  

(3)

This equation is more generally known as head conduction equation (without any heating source).
A general solution to equation (3) is well established. It is of the form

\[
n = \frac{1}{(\sqrt{4Dt})^l} \int \cdots \int \varphi(\xi_1 \cdots \xi_l) e^{- \frac{(x_1^2 + \cdots + x_l^2)}{4Dt} - \frac{(x_1 - \xi_1)^2 + \cdots + (x_l - \xi_l)^2}{4Dt}} d\xi_1 \cdots d\xi_l
\]  

(4)

where \( \varphi \) is the distribution of \( n(x_1, \ldots, x_l) \) at \( t=0 \) [Mathematics Handbook, p. 720]. One easy solution can be found when \( \varphi = A \left( \frac{x_1^2 + \cdots + x_l^2}{4Dt_o} \right)^l e^{- \frac{x_1^2 + \cdots + x_l^2}{4Dt_o}} \)  

(5)

With the above initial condition, we have

\[
n = \frac{1}{(\sqrt{4Dt})^l} \int \cdots \int A \left( \frac{x_1^2 + \cdots + x_l^2}{4Dt_o} \right)^l e^{- \frac{(x_1 - \xi_1)^2 + \cdots + (x_l - \xi_l)^2}{4Dt}} d\xi_1 \cdots d\xi_l
\]  

(6)

The solution to this equation is

\[
n = \frac{A}{(\sqrt{4D(t + t_o)})^l} e^{- \frac{x_1^2 + \cdots + x_l^2}{4D(t + t_o)}}
\]  

(7)

If \( t_o \) is large compared to the time-scale of interest, the decay constant in a multi-dimensional space is the same as in an one-dimensional space. The constant, \( A \), can be determined by integrating the initial distribution.

References:
