THEORY OF MEASUREMENTS

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OUTLINE

• Antenna-Sky Coupling
• Noise
  – the Radiometer Equation
  – Minimum Tsys
  – Performance measures
• System Response
  – Gain & linearity
  – quantization
Antenna-Sky Coupling

Power Received:

\[
P_{\text{rec}}(\nu) = \frac{1}{2} A_e \int_\Omega I_\nu(\theta, \varphi) P_n(\theta, \varphi) \, d\Omega \text{ watts Hz}^{-1}
\]

Effective Area \(^\wedge\)  

\[
A_e = \eta_{\text{aperture}} A_{\text{geometrical}}
\]

Source \(^\wedge\)  

Specific intensity (surface brightness)

\[
\int P_n(\theta, \phi) d\Omega = \Omega_{\text{Ant}}
\]

Factor \(\frac{1}{2}\) comes from one of two polarizations.
Antenna-Sky Coupling

Power Received:

\[ P_{\text{rec}}(\nu) = \frac{1}{2} A_e \int_{\Omega} I_\nu(\theta, \varphi) P_n(\theta, \varphi) \, d\Omega \]  \text{ watts Hz}^{-1}

\[ I_\nu(\theta, \varphi) = \frac{2h\nu}{\lambda^2} \left( e^{\frac{h\nu}{kT(\theta, \varphi)}} - 1 \right) \rightarrow \frac{2h\nu}{\lambda^2} \frac{kT(\theta, \varphi)}{\nu} = \frac{2kT(\theta, \varphi)}{\lambda^2} \text{ Long wavelength (Rayleigh-Jeans)} \]

Use Antenna Theorem:

\[ A_{\text{eff}} \Omega_{\text{Ant}} = \lambda^2 \]

And express result in terms of Antenna temperature

\[ T_{\text{Ant}} = \frac{P_\nu}{k} \]

\[ T_{\text{Ant}} = \frac{1}{\Omega_{\text{Ant}}} \int T(\theta, \varphi)P_n(\theta, \varphi) \, d\Omega \]

Antenna temperature is the average BB brightness temp. over the whole Beam pattern.
\[ T_{\text{Ant}} = \frac{1}{\Omega_{\text{Ant}}} \int T(\theta, \phi) P_n(\theta, \phi) d\Omega \]

Compact, Isothermal BB source:
\[ T_{\text{Ant}} = \frac{\Omega_{\text{Src}}}{\Omega_{\text{Ant}}} T_{\text{Src}} \]

Main-beam filling isothermal BB source:
\[ T_{\text{Ant}} = \frac{\Omega_{\text{MB}}}{\Omega_{\text{Ant}}} T_{\text{Src}} = \eta_B T_{\text{Src}} \]

Very compact, non-thermal sources better described in terms of flux density:
\[ T_{\text{Ant}} = \frac{A_{\text{eff}}}{2k} S_{\text{Src}} \]

More collecting area is a big win for very compact sources.
For Imaging sometimes you want a big dish, sometimes you want a smaller dish.

Ideal value = 1
Determined by optics & telescope illumination.
The Antenna as a Spatial Filter

Imagine making a map by moving the telescope around and recording the antenna temperature at each \( \text{(nyquist)} \) point. In 1-D:

\[
T_A(\theta) = \frac{1}{\Omega_A} \int T_\nu(\theta') \tilde{P}_n(\theta - \theta') \, d\theta'
\]

\[
T_A(\theta) = T_\nu(\theta) \ast P'(\theta)
\]

The Convolution Theorem gives

\[
\tilde{T}_A(u) = \tilde{T}_\nu(u) \times \tilde{P}'(u)
\]

where \( \tilde{T}_A(u) \leftrightarrow T_A(\theta) \), etc.
The Antenna as a Spatial Filter

Imagine making a map by moving the telescope around and recording the antenna temperature at each (nyquist) point. In 1-D:

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The Convolution Theorem gives

\[ \tilde{T}_A(u) = \tilde{T}_\nu(u) \times \tilde{P}'(u) \]

... which is also the square of FT of Antenna "illumination"

where \( \tilde{T}_A(u) \leftrightarrow T_A(\theta) \), etc.
Antenna Pattern and its Fourier Transform

Figure 3. The beam pattern and spatial frequency response of a uniformly illuminated one-dimensional aperture of length $L$. 
Observing Strategy & Spatial Filtering

The real world introduces contaminating signals which must be removed. This requirement often drives the choice of *observing strategy*.

Beam Switching/Chopping → *differential* sky image.

Emerson, Klein & Haslam (EKH) described how to account for this.
\[ T_{\text{Ant}}(x) = T_{\text{Sky}}(x) \otimes (B(x) \otimes D(x)) \]
Tycho SNR w/Effelsburg 100m

Single-beam sky map
Tycho SNR w/ Effelsburg 100m

"Naïve" Dual-beam Beamswitched sky map
Tycho SNR w/Effelsburg 100m

Dual-beam
Beamswitched sky map
After EKH
Variations, other Approaches

- **EKH-2**
  - Use multiple chops to sample missing spatial frequencies

- **Least Squares Map Making**
  
  \[ \vec{d} = A\vec{m} \]

  \[ \vec{m} = (A^T A)^{-1} A^T \vec{d} \]

  - Very general; suitable for use with focal plane arrays
  - Used by WMAP and other CMB experiments
  - E.g., Fixsen, Moseley & Arendt (2000)
What is the intrinsic noise in the signal going into the telescope? Consider the case that we’re looking at a grey body.

\[ (\Delta n_{\text{rms}})^2 = n^2 + n \]

Bose (wave noise) term Dominates in long wavelength (RJ) regime.

\[ n = \frac{\varepsilon}{e^{\frac{h\nu}{kT}} - 1} \]

At low count rates photon arrival times are uncorrelated (Poisson / shot noise)

\( n \): # photons/sec/Hz
\( \varepsilon \): emissivity

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Noise

What is the intrinsic noise in the signal going into the telescope? Consider the case that we’re looking at a grey body.

\[ (\Delta n_{\text{rms}})^2 = n^2 + n \]

\[ n = \frac{\varepsilon}{e^{h\nu/kT} - 1} \]

- \( n \): # photons/sec/Hz
- \( \varepsilon \): emissivity

Atmosphere: \( T \sim 300 \text{K} \)

\( \nu = 1 \text{ GHz}, \varepsilon = 0.01 \) (radio)
- \( n \sim 60 \)

\( \nu = 200 \text{ GHz}, \varepsilon = 0.1 \) (mm)
- \( n \sim 3 \)

\( \nu = 1 \text{ THz}, \varepsilon = 0.05 \) (submm balloon)
- \( n \sim 0.3 \)
What is the intrinsic noise in the signal going into the telescope? Consider the case that we’re looking at a grey body.

\[
(\Delta n_{rms})^2 = n^2 + n \]

\[
n = \frac{\varepsilon}{e^{\frac{h\nu}{kT}} - 1}
\]

\[
\Delta T = \frac{T}{\sqrt{\Delta vT}}
\]

\[
T \rightarrow T_{object} + T_{RX} + T_{atmosphere} + T_{spillover} \equiv T_{Sys}
\]

- proportional to input signal (not Square Root)
- Other contributors to the signal add linearly to an overall System temperature.
- Typical Tsys’s: few 10s of K
- 100 MHz BW reduce the noise by 10,000 in 1 sec
  - few mK RMS in 1 sec
Minimum $T_{sys}$ for a Coherent Amplifier

"Coherent Amplifier": phase-preserving

$$\Delta E \Delta t > \hbar \quad \rightarrow \quad \Delta n \Delta \phi > 1$$

\[ n_2 = G n_1 \]
\[ \phi_2 = \phi_1 \]
Minimum $T_{\text{sys}}$ for a Coherent Amplifier

"Coherent Amplifier": phase-preserving

$$\Delta E \Delta t > \hbar \quad \rightarrow \quad \Delta n \Delta \phi > 1$$

$$\Delta n_1 = \frac{\Delta n_2}{G}$$

Problem is fixed by assuming the amplifier adds ~one photon per hz per second uncertainty to the measurement of $n_1$

$$T_{RX,\text{min}} = \frac{h \nu}{k}$$
Minimum $T_{sys}$ for a Coherent Amplifier

- 1 GHz: 0.05 K << BG
- 100 GHz: 4.8 K < BG
- Optical: 10,000 K >> BG

\[ T_{RX,min} = \frac{h \nu}{k} \]

*these are theoretical minimums ... real systems often noisier

If phase is not preserved- “direct detection” - this limit does not exist (Bolometers, optical CCD cameras, etc.)

Your measurement can in principle be limited by only the noise in the input photon field [BLIP]

Note: even “photon counting” systems in the radio will not have poisson statistics; they will obey the Radiometer Equation.
Performance Measures

From before:  \[ T_{\text{Ant}} = \frac{A_{\text{eff}}}{2k} S_{\text{Src}} \rightarrow \Gamma \equiv \frac{T_{\text{Ant}}}{S_{\text{Src}}} = \frac{A_{\text{eff}}}{2k} \] Gain

An effective aperture of 2760 m² is required to give a sensitivity of 1.0 K/Jy.

\[ \text{SNR} = \frac{T_{\text{Ant}}}{\Delta T} \propto \frac{\Gamma}{T_{\text{Sys}}} \] Units of 1/Janskys OR [meters²/Kelvin] (large is good)
You sometimes see the System-Equivalent Flux Density, SEFD (small is good)

“Mapping Speed” commonly defined as:

\[ \text{Mapping Speed} = \frac{\text{Area}}{(\text{noise})^2(\text{time})} \]

These are single-pixel measures. For mapping, increase by Nfeeds.
Gain Effects: deviations from linearity

Integrated power is what usually matters (RFI)
Gain Effects: deviations from linearity

Be aware of the limitations of the instrument. You’re using & calibrate at a similar total power level to what your science observations will see.

Integrated power is what usually matters (RFI)
Gain Effects: fluctuations

Gain drifts over the course of an observing Session(s) easily removed with instrumental Calibrators (e.g., noise diodes)

• when gains or input power change, attenuators often need to be changed & calibration is then needed also.

Short term gain fluctuations can be more problematic (see continuum lecture)
Sampling/Quantization & Dynamic Range: Postdetection

Diode or square law detector:
   Turns E-field into some output
   Proportional to \( (E^2) \)

Common for continuum systems.

Robust & simple (large dynamic range)
\[ \approx 2^N \Delta P \]

A/D
2\(^N\) levels
(N~14)

\[ \Delta P = \frac{P}{\sqrt{\Delta v \tau}} \]

~1 level
A/D sample time
Sampling/Quantization & Dynamic Range: Predetection

Sample the E-field itself.

Samplers must be much faster and sample much more coarsely (typically just a few levels)

Dynamic range limitations more important
   One usually requires variable attenuators to get it right (“Balancing”).

Small # of levels increases the noise level.
   (K-factor in Radiometer equation)

More levels → greater dynamic range (RFI robustness), greater sensitivity

Monitor levels through your observation.
THANKS

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Further reading:
• Tools of Radio Astronomy (Rholfs & Wilson)
• Radio Astronomy (Krauss)
• Synthesis Imaging in Radio Astronomy II (Tayloer, Carilli & Pereley)
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*S. Stanimirović, D. R. Altschuler, P. F. Goldsmith, and C. J. Salter*

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