A HEURISTIC INTRODUCTION TO RADIOASTRONOMICAL POLARIZATION

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STOKES PARAMETERS: BASICS

Stokes parameters are linear combinations of power measured in *orthogonal polarizations*.

$$I = E_X^2 + E_Y^2 = E_{0^\circ}^2 + E_{90^\circ}^2$$
$$Q = E_X^2 - E_Y^2 = E_{0^\circ}^2 - E_{90^\circ}^2$$
$$U = E_{45^\circ}^2 - E_{-45^\circ}^2$$
$$V = E_{LCP}^2 - E_{RCP}^2$$

The first is total intensity. It is the sum of *any two orthogonal* polarizations.

The second two completely specify linear polarization.

The last completely specifies circular polarization.

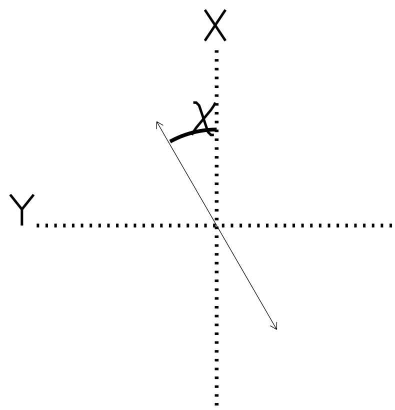
We like to write the *Stokes vector*

$$\mathbf{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} .$$

CONVENTIONAL LINEAR POLARIZATION PARAMETERS

AS χ CHANGES, WE HAVE

$$\frac{Q}{I} = p_{QU}\cos(2\chi)$$
$$\frac{U}{I} = p_{QU}\sin(2\chi)$$



POSITION ANGLE OF LINEAR POLARIZATION:

$$\chi = 0.5\tan^{-1}\frac{U}{Q}$$

FRACTIONAL LINEAR POLARIZATION:

$$p_{QU} = \left[\left(\frac{Q}{I}\right)^2 + \left(\frac{U}{I}\right)^2\right]^{1/2}$$

OTHER CONVENTIONAL POLARIZATION PARAMETERS

FRACTIONAL CIRCULAR POLARIZATION:

$$p_V = \frac{V}{I}$$

TOTAL FRACTIONAL POLARIZATION:

$$p = \left[\left(\frac{Q}{I} \right)^2 + \left(\frac{U}{I} \right)^2 + \left(\frac{V}{I} \right)^2 \right]^{1/2}$$

If both p_{QU} and p_V are nonzero, then the polarization is *elliptical*.

RADIOASTRONOMICAL FEEDS

Feeds are normally designed to approximate pure linear or circular—known as *native linear* **or** *native circular***.**

Generally speaking, native linear feeds are intrinsically accurate and provide true linear. However, *native circular feeds are less accurate and their exact polarization response is frequency dependent***.**

At the *GBT*:

- **Feeds below 8 GHz are native linear.**
- **Feeds above 8 GHz are native circular. For the 8-10 GHz receiver, the response changes from pure circular at 8 GHz to 14% elliptical at 10 GHz.**

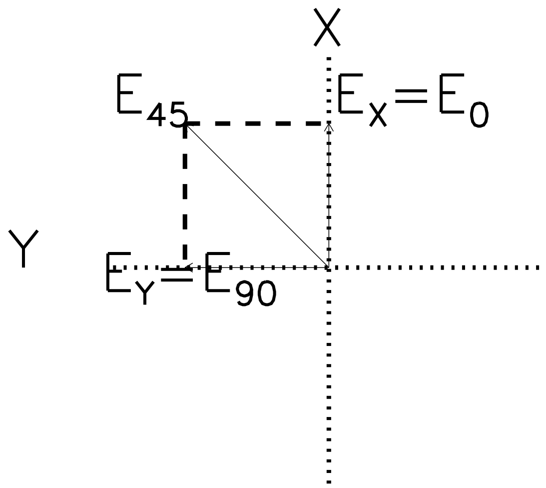
At *ARECIBO*:

- **Feeds at 1-2 GHz and 4-6 GHz are native linear.**
- **most others are native circular, achieved with waveguide turnstile junctions with very accurate polarization at the center frequency. However, these are narrow band devices: the feeds become increasingly elliptical, changing to linear and back again over frequency intervals ~ 100 MHz!**

REAL RADIO ASTRONOMERS MEASURE ALL STOKES PARAMETERS SIMULTANEOUSLY!

Extracting two orthogonal polarizations provides *all* the information; you can synthesize *all* other E fields from the two measured ones!

Example: Sample (E_X, E_Y) and synthesize E_{45} from (E_X, E_Y):



To generate E_{45}, add (E_X, E_Y) with no phase difference.

To generate E_{LCP}, add (E_X, E_Y) with a 90° phase difference.

CARRYING THROUGH THE ALGEBRA FOR THE TWO LINEARS ...

It's clear that

$$E_{45^\circ} = \frac{E_{0^\circ} + E_{90^\circ}}{\sqrt{2}}$$

$$E_{-45^\circ} = \frac{E_{0^\circ} - E_{90^\circ}}{\sqrt{2}}$$

Write the two linear Stokes parameters:

$$Q = E_X^2 - E_Y^2 = E_{0^\circ}^2 - E_{90^\circ}^2$$

$$U = E_{45^\circ}^2 - E_{-45^\circ}^2 = 2E_X E_Y$$

STOKES U IS GIVEN BY THE CROSS-CORRELATION $\boldsymbol{E_X E_Y}$

To get V, throw a 90° phase factor into the correlation.

DOTTING THE I'S AND CROSSING THE T'S GIVES...

Carrying through the algebra and paying attention to complex conjugates and extracting the real part of the expressions yields (for the case of sampling linear polarization (X, Y):

$$I = E_X\overline{E_X} + E_Y\overline{E_Y} \equiv \mathbf{XX}$$

$$Q = E_X\overline{E_X} - E_Y\overline{E_Y} \equiv \mathbf{YY}$$

$$U = E_X\overline{E_Y} + \overline{E_X}E_Y \equiv \mathbf{XY}$$

$$iV = E_X\overline{E_Y} - \overline{E_X}E_Y \equiv \mathbf{YX}$$

The overbar indicates the complex conjugate. These products are time averages; we have omitted the indicative $\langle\rangle$ brackets to avoid clutter.

IMPORTANT FACT:

Stokes Q **and** U **are sums and differences of** *self-products***.**

Stokes U **and** V **are sums and differences of** *cross products***.**

For small polarizations, *cross* **products are** *much less subject to error* **than self-products.**

COROLLARY: To accurately measure small *linear* **polarization, use a dual** *circular* **feed; to accurately measure small** *circular* **polarization, use a dual** *linear* **feed.**

THE MUELLER MATRIX

The output terminals of a native linear feed provides voltages that sample the E-fields E_X and E_Y; this it provides directly the Stokes parameters I (from $\mathbf{XX} + \mathbf{YY}$) and Q (from $\mathbf{XX} - \mathbf{YY}$).

Rotating the feed by 45° interchanges XX and YY, so it provides directly I and U.

A native circular feed adds a 90° phase to X (or Y) and provides directly the Stokes parameters I (from $\mathbf{XX}+\mathbf{YY}$) and V (from $\mathbf{XX}-\mathbf{YY}$). We express these interchanges of power among Stokes parameters with the ***Mueller matrix*** **M.**

$$\begin{bmatrix} \mathbf{XX} \\ \mathbf{YY} \\ \mathbf{XY} \\ \mathbf{YX} \end{bmatrix} = \mathbf{M} \cdot \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} \tag{1}$$

Some examples of Mueller matrices:

(1) A dual linear feed: M is unitary.

(2). A dual linear feed rotated 45°**:** Q **and** U **interchange, together with a sign change as befits rotation:**

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{2}$$

(3). A dual linear feed rotated 90°**, which reverses the signs of** Q **and** U**:**

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{3}$$

(4) The above are special instructive cases. As an alt-az telescope tracks a source, the feed rotates on the sky by the *parallactic angle* PA_{az}**.**

$$\mathbf{M_{SKY}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2PA_{az} & \sin 2PA_{az} & 0 \\ 0 & -\sin 2PA_{az} & \cos 2PA_{az} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{4}$$

The central 2×2 **submatrix is, of course, nothing but a rotation matrix.** $\mathbf{M_{SKY}}$ **doesn't change** I **or** V**.**

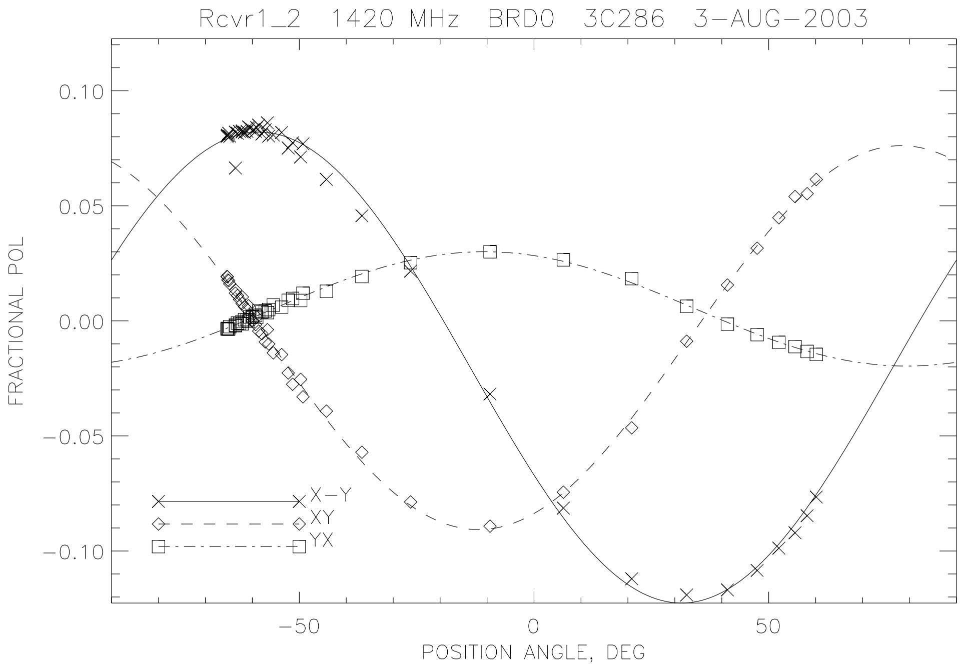
(5). A dual circular feed, for which $V = \mathbf{XX} - \mathbf{YY}$**:**

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} . \tag{5}$$

For a perfect system, as we track a linearly polarized source across the sky the parallactic angle PA changes. This should produce:

- **For XX − YY,** $[\cos 2(PA_{AZ} + PA_{SRC})]$ **centered at zero;**
- **For XY,** $[\sin 2(PA_{AZ} + PA_{SRC})]$ **centered at zero;**
- **For YX, zero (most sources have zero circular polarization).**

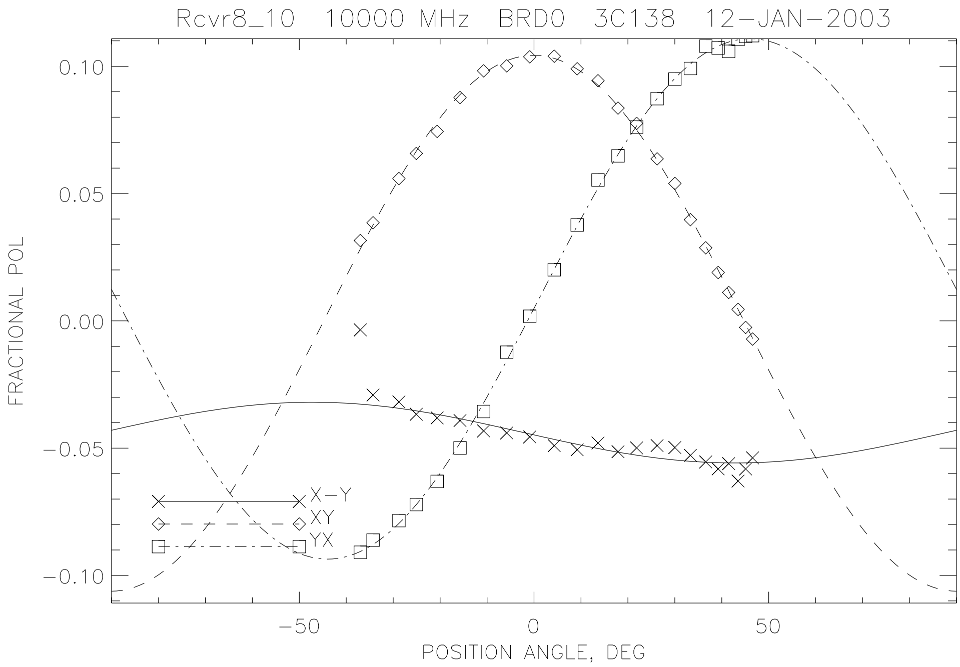
REAL DATA: NATIVE LINEAR POLARIZATION




```
DELTAG = −0.041 +/-  0.017
PSI =  −16.6 +/-    5.0
ALPHA =    0.3 +/-    2.5
EPSILON =  0.004 +/-  0.004
PHI =  161.0 +/-   53.1
QSRC = −0.041 +/-  0.006
USRC = −0.085 +/-  0.006
POLSRC =  0.095 +/-  0.000
PASRC (**UNCORRECTED FOR M_ASTRO**) =  −57.7 +/-    0.0
NR GOOD POINTS=       41      42      42/      42
SCAN 26
-------------------------------------------------------
MUELLER MATRIX:
```

1.0000	−0.0203	−0.0085	0.0027
−0.0203	1.0000	0.0002	0.0097
−0.0073	−0.0028	0.9586	0.2849
0.0052	−0.0093	−0.2849	0.9585

REAL DATA: NATIVE CIRCULAR POLARIZATION




```
DELTAG = −0.088 +/−  0.004
PSI =    0.0 +/−    0.0
ALPHA =  −48.3 +/−    0.5
EPSILON =  0.004 +/−  0.001
PHI =   96.1 +/−   13.5
QSRC = −0.002 +/−  0.001
USRC =  0.104 +/−  0.001
POLSRC =  0.104 +/−  0.000
PASRC (**UNCORRECTED FOR M_ASTRO**) =   45.5 +/−    0.0
NR GOOD POINTS=       22      23      23/      23
SCAN 329
--------------------------------------------------------------
MUELLER MATRIX:

      1.0000      0.0136     −0.0009      0.0426
     −0.0439     −0.1154      0.0000     −0.9933
     −0.0009     −0.0000      1.0000      0.0000
      0.0086      0.9934      0.0000     −0.1151
```

THE SINGLE MATRIX FOR THE RADIOASTRONOMICAL RECEIVER

The observing system consists of several distinct elements, each with its own Mueller matrix. The matrix for the whole system is the product of all of them. Matrices are not commutative, so we must be careful with the order of multiplication.

$$\mathbf{M_{TOT}} = \begin{bmatrix} 1 & (-2\epsilon\sin\phi\sin 2\alpha + \frac{\Delta G}{2}\cos 2\alpha) & 2\epsilon\cos\phi & (2\epsilon\sin\phi\cos 2\alpha + \frac{\Delta G}{2}\sin 2\alpha) \\ \frac{\Delta G}{2} & \cos 2\alpha & 0 & \sin 2\alpha \\ 2\epsilon\cos(\phi+\psi) & \sin 2\alpha\sin\psi & \cos\psi & -\cos 2\alpha\sin\psi \\ 2\epsilon\sin(\phi+\psi) & -\sin 2\alpha\cos\psi & \sin\psi & \cos 2\alpha\cos\psi \end{bmatrix}.$$

NOTE: The Mueller matrix has 16 elements, but *ONLY 7 INDEPENDENT PARAMETERS*. **The matrix elements are not all independent.**

ΔG is the error in relative intensity calibration of the two polarization channels. It results from an error in the relative cal values (T_{calA}, T_{calB}).

ψ is the phase difference between the cal and the incoming radiation from the sky (equivalent in spirit to $L_X - L_Y$ on our block diagram..

α is a measure of the voltage ratio of the polarization ellipse produced when the feed observes pure linear polarization.

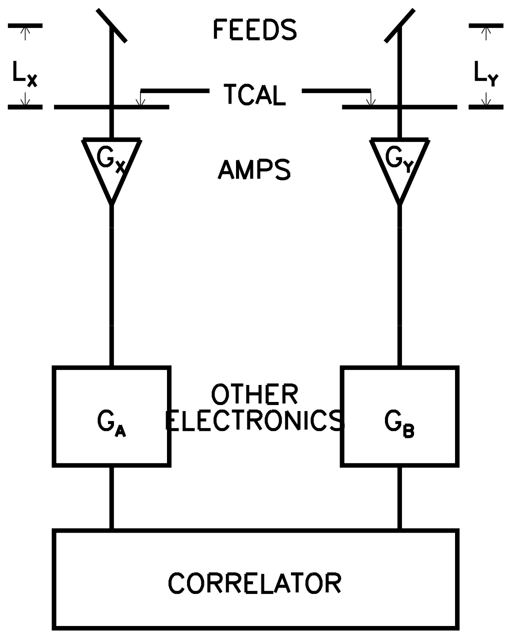
χ is the relative phase of the two voltages specified by α.

ϵ is a measure of imperfection of the feed in producing nonorthogonal polarizations (false correlations) in the two correlated outputs.

ϕ is the phase angle at which the voltage coupling ϵ occurs. It works with ϵ to couple I with (Q, U, V).

θ_{astron} is the angle by which the derived position angles must be rotated to conform with the conventional astronomical definition.

THE RECEIVER INTRODUCES INSTRUMENTAL POLARIZATION



Cable lengths introduce phase delays. Cables are never identical!

Amplifier gains are *complex*: amplitude *and* phase.

Most amplifiers introduce a 180° phase inversion. If the two channels have different numbers of amplifiers. . .

The receiver introduces mutual coupling among all four Stokes parameters. This coupling is described:

• **For the 2-element voltage (Jones) vector, by the** 2×2 *Jones Matrix*;

• **For the 4-element Stokes vector, by the** 4×4 *Mueller Matrix*.

One must correct for this!

THE CAL...OUR CONVENIENT INTENSITY AND PHASE REFERENCE

The signal and cal share common paths after the wavy line.

If we know the relative gains and phase of the cal, we remove the system contributions.

We determine $(G, \phi)_{cal}$ by comparing the cal deflection with the deflection of a known astronomical source (e.g. 3C286). This determines the Mueller matrix parameters.

We assume that the cal remains constant. After returning the to the telescope after an interval of months, it is wise to check!

AN IMPORTANT POINT: THETA VARIES WITH FREQUENCY

The difference in path length ΔL means that the relative phase of E_x and E_y changes with frequency. Thus θ changes with frequency.

$$\frac{d\theta}{df} = \frac{2\pi\ \Delta L}{c} \approx 0.3\ \frac{\text{rad}}{\text{MHz}}$$

This corresponds to

$$\Delta L \approx 20\ \text{m}$$

The consequences:

- **You must include the frequency dependence in your least squares fit. This is a bit tricky.**
- **You cannot do continuum polarization over significant bandwidths without including $\frac{d\theta}{df}$.**

REMEMBER THIS # 1: AVERAGING LINEAR POLARIZATIONS!!!

Suppose you average two polarization observations together: Observation 1 has $p = 13.6\%$ **and** $\chi = 2^\circ$

Observation 2 has $p = 13.7\%$ **and** $\chi = 178^\circ$

NOTE THAT THE POSITION ANGLES AGREE TO WITHIN 4 DEGREES.

If you average p **and** χ**, you get** $p = 13.65\%$ **and** $\chi = 90^\circ$**.**

THIS IS INCORRECT!!!!!!!!!

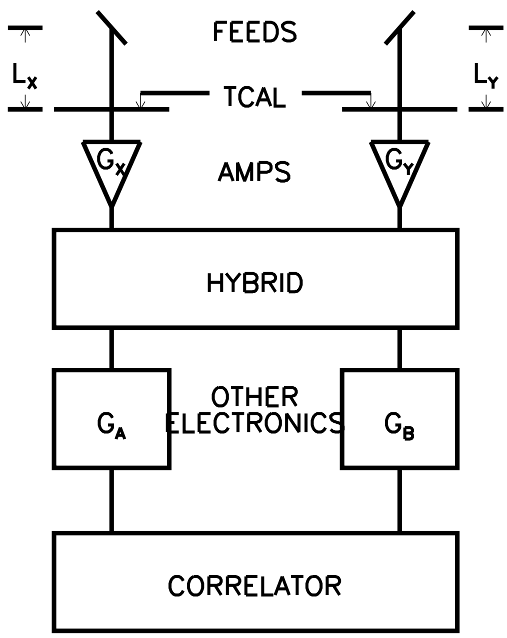
Often you find yourself needing to combine polarizations. For example, if you measure the polarization of some object several times, you need to average the results.

There is only one *proper* **way to combine polarizations, and that is to use the Stokes parameters. The reason is simple: because of conservation of energy, powers add and subtract.**

What you must always do is convert the fractional polarization and position angle to Stokes parameters, average the Stokes parameters, and convert back to fractional polarization and position angle.

REMEMBER THIS # 2: SHOULD YOU GENERATE CIRCULARS WITH A POST-AMP HYBRID???

Some astronomers believe that source fluxes (that's Stokes I) are better measured with circular polarization. Receiver engineers accommodate them with a post-amp hybrid:

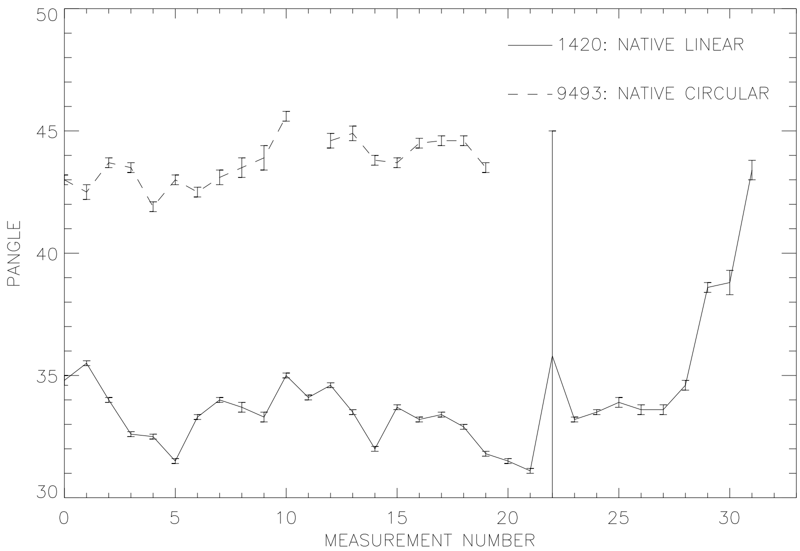


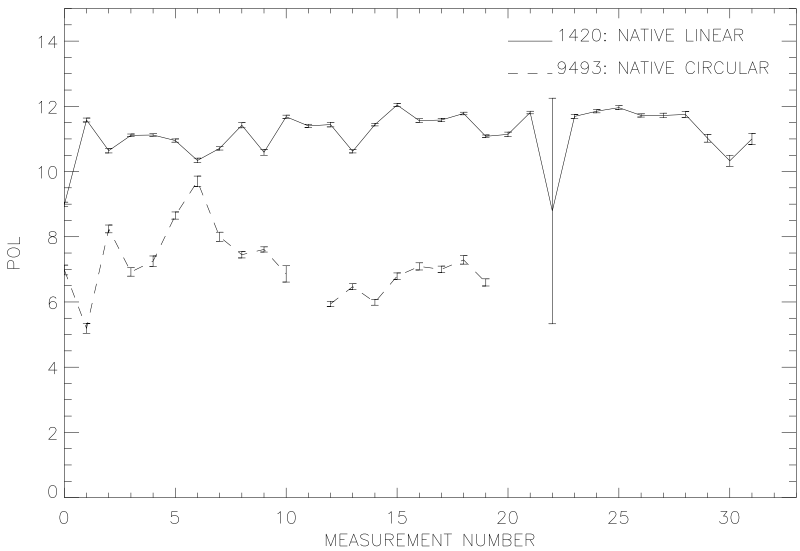
Using this system properly requires a much more complicated calibration procedure.

Example: If $G_Y = 0$, the system still appears to work!!

JUST SAY NO!!!!!!

REMEMBER THIS #3: CROSSCORRERLATION VERSUS DIFFERENCING!!!

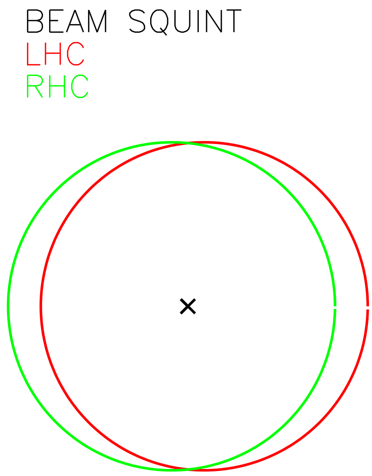


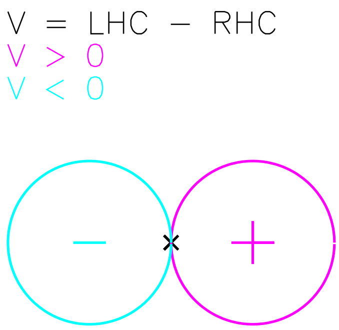


Crosscorrelation (9493 MHz) gives better results than differencing (1420 MHz) for posi-

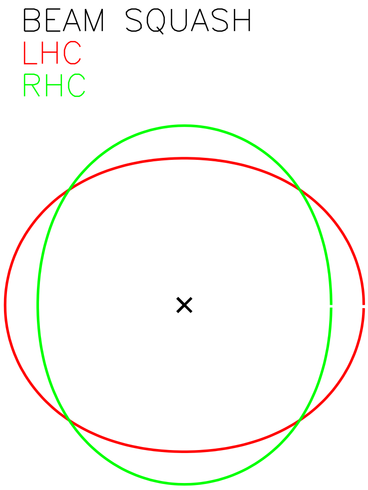
tion angle measurements. You *NEVER* **see wild fluctuations with crosscorrelation, while they are common with differencing. (At 1420 MHz 3C286 is** $\sim 1.5T_{sys}$**; at 9493 MHz 3C286 is** $\sim 0.5T_{sys}$**). (POL for 9493 MHz is displaced downward by 4 for clarity).**

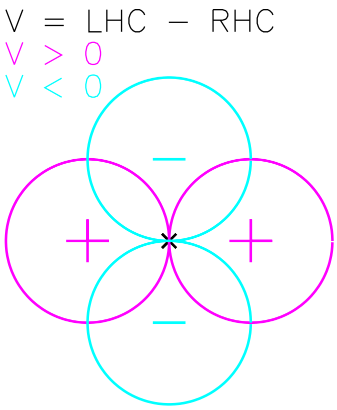
POLARIZED BEAM EFFECTS: BEAM SQUINT



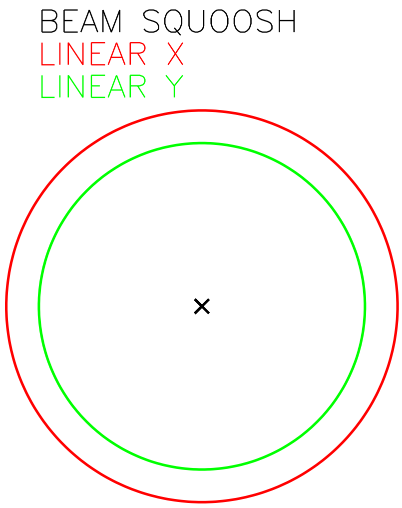


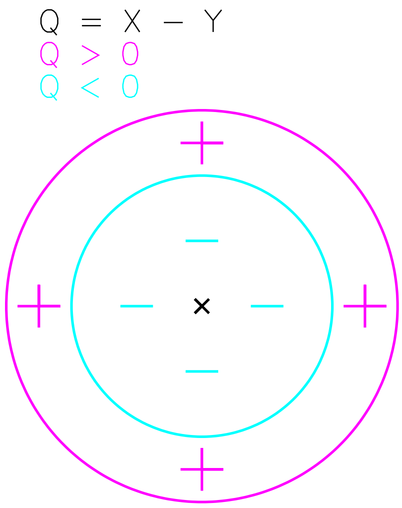
POLARIZED BEAM EFFECTS: BEAM SQUASH

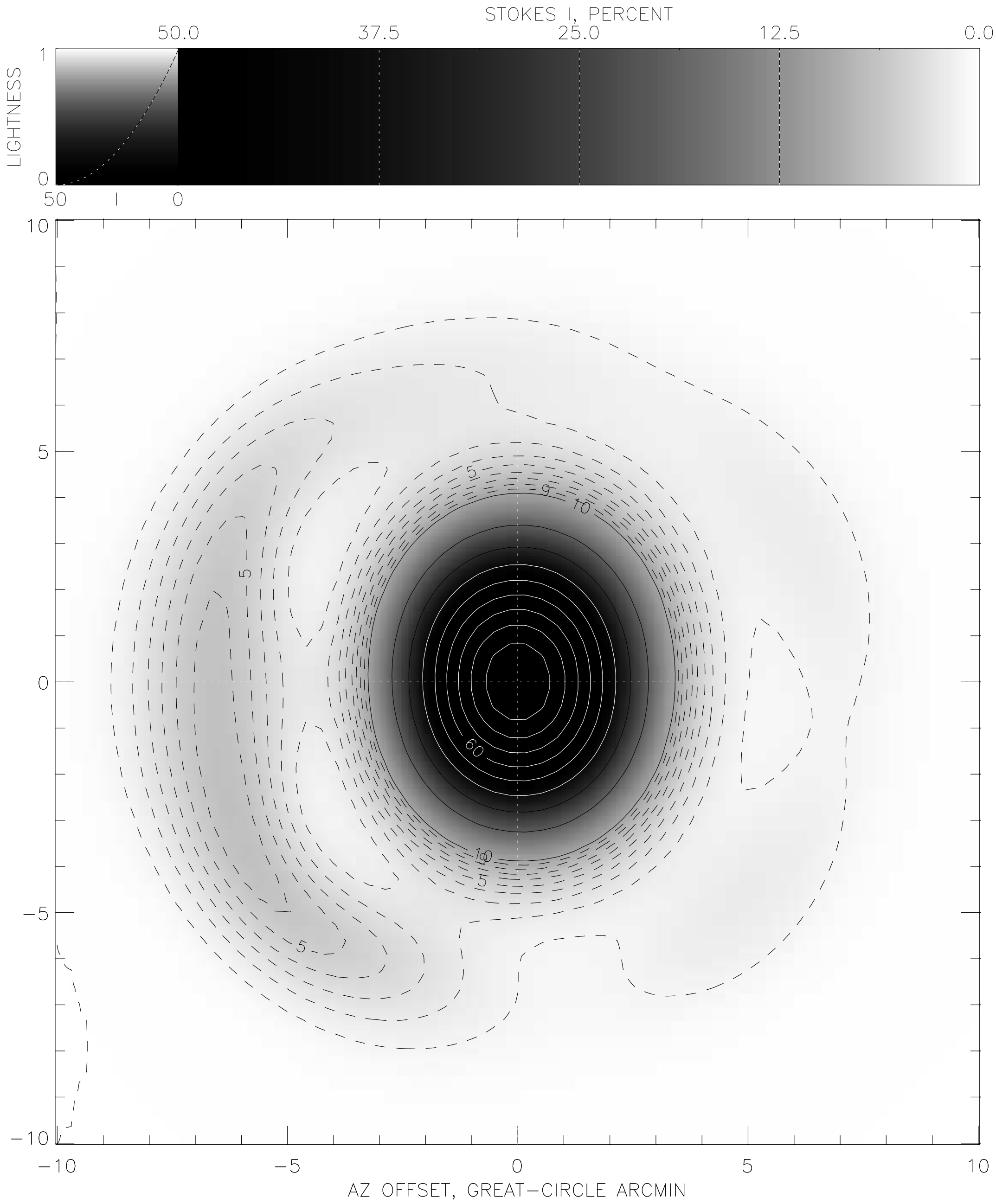


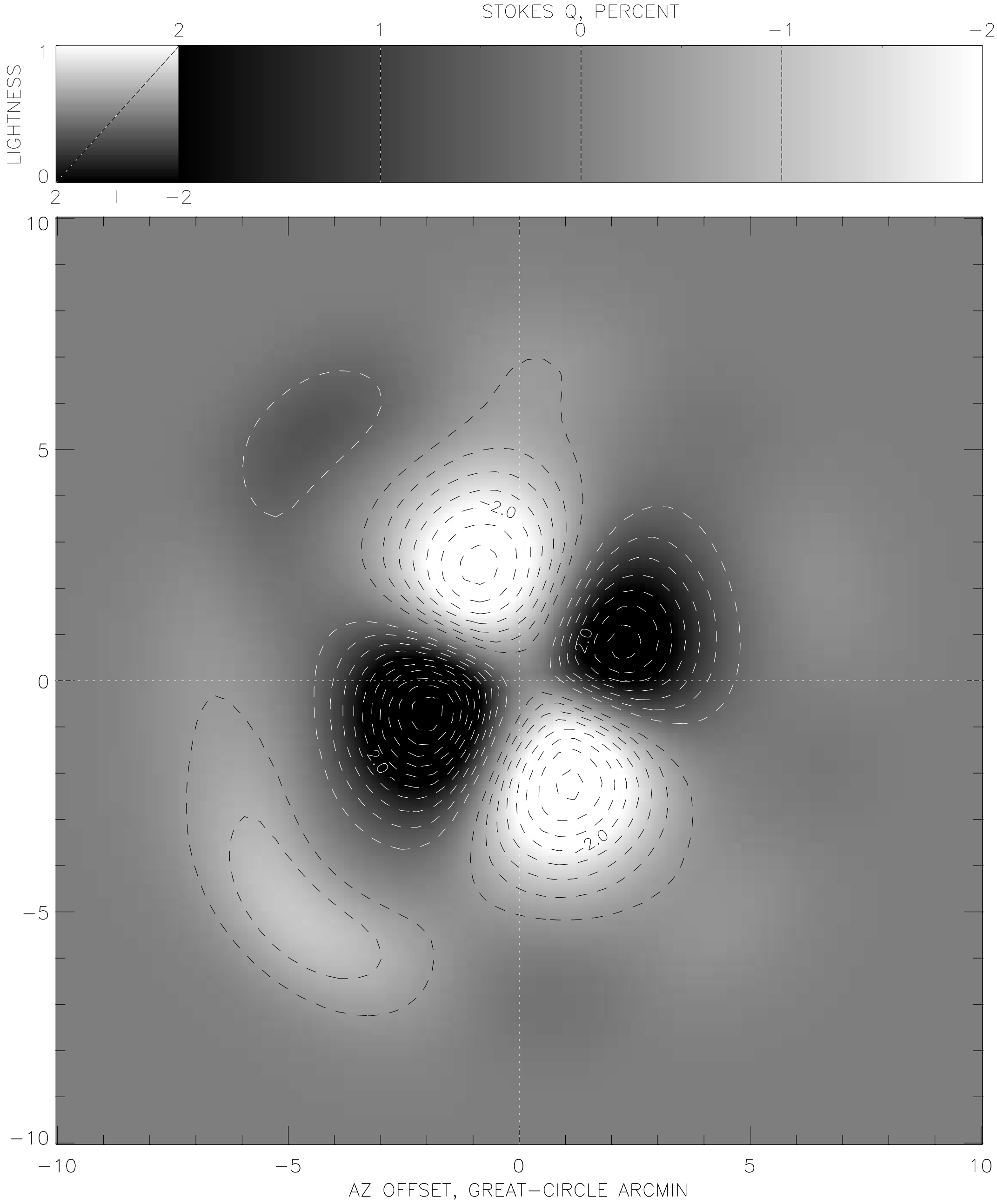


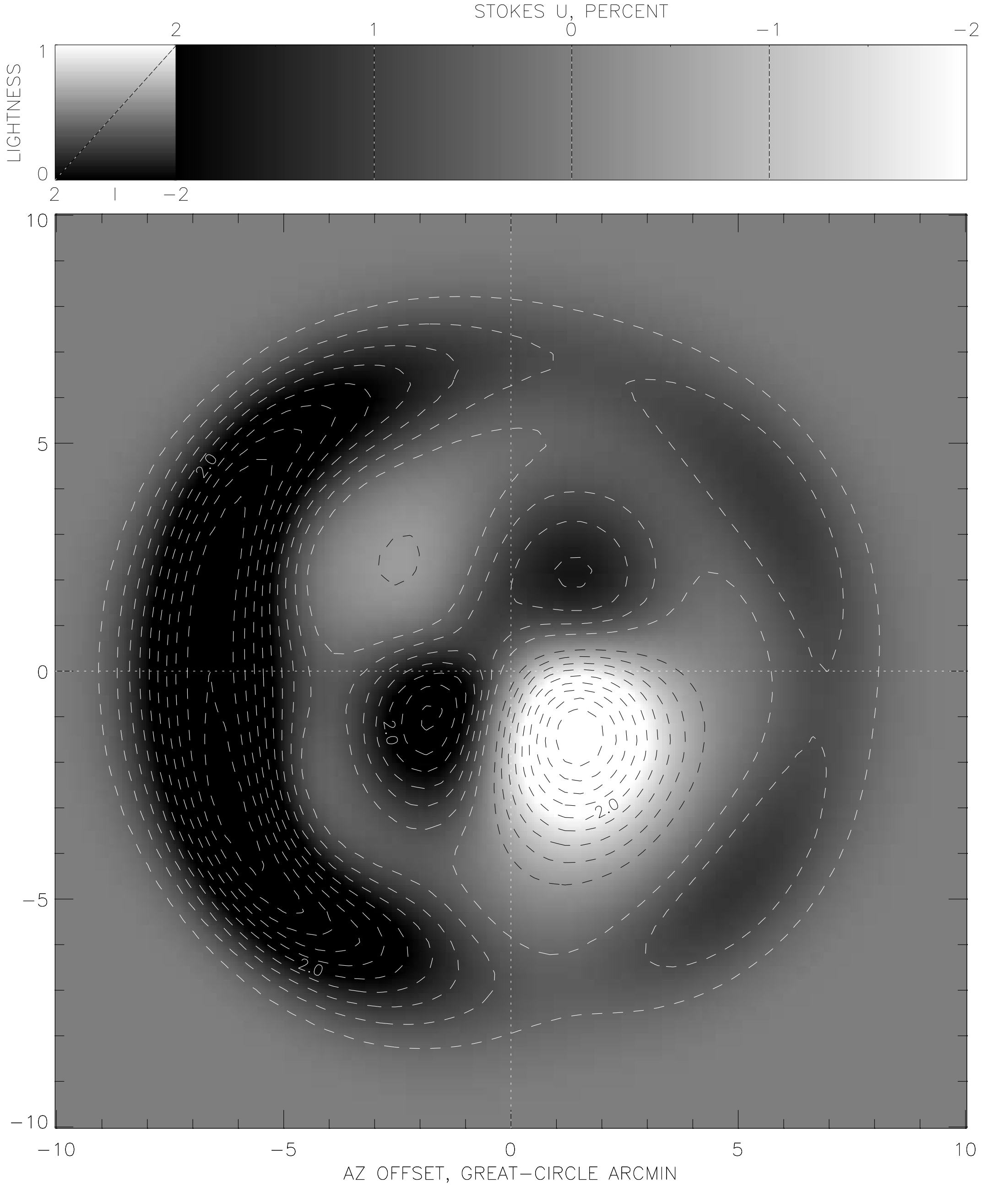
POLARIZED BEAM EFFECTS: BEAM SQUOOSH

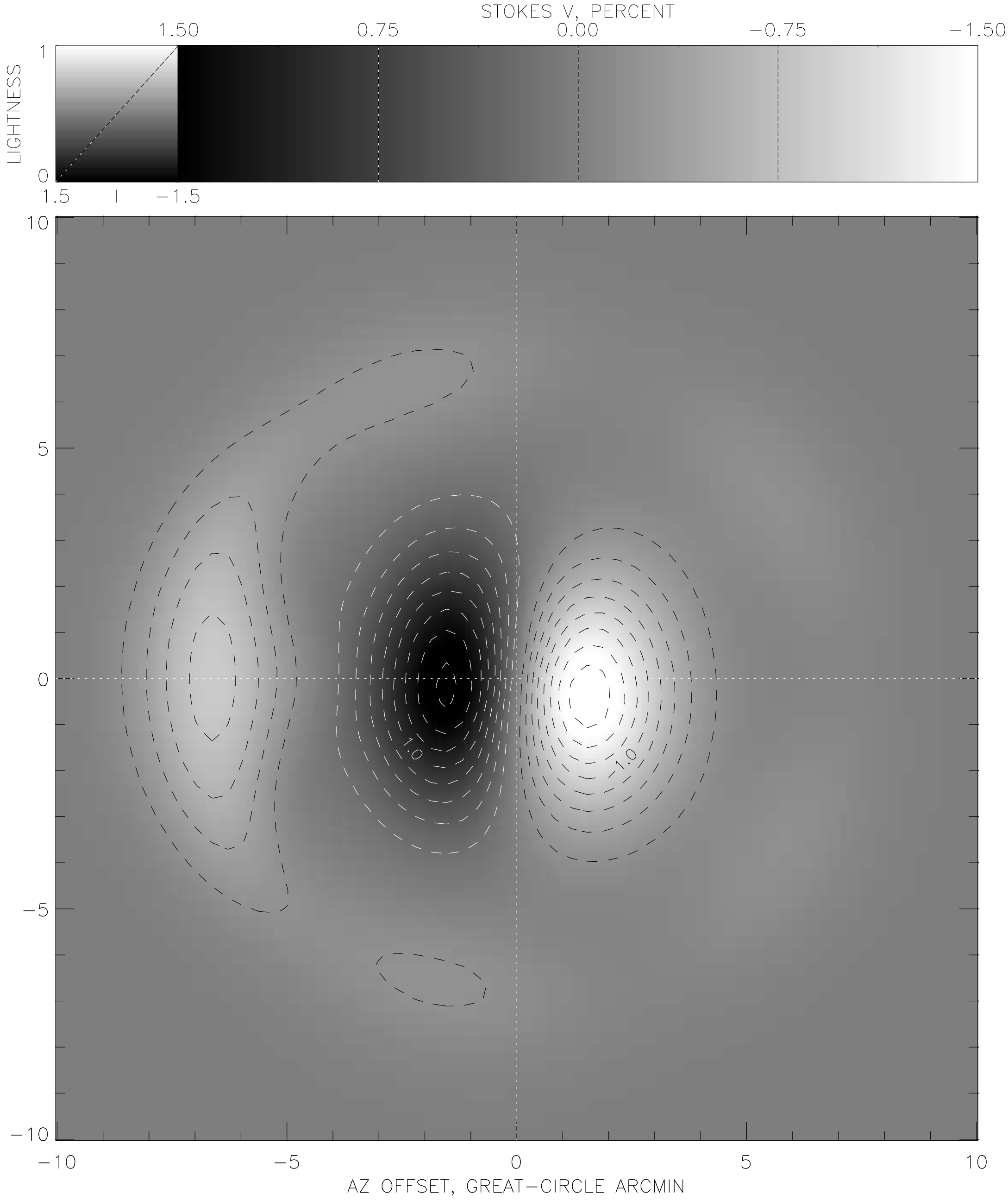





STOKES I, PERCENT
50.0
37.5
25.0
12.5
0.0
LIGHTNESS
1
0
50
I
0
ZA OFFSET, ARCMIN
AZ OFFSET, GREAT−CIRCLE ARCMIN
10
5
0
−5
−10
5
9
10
60


STOKES Q, PERCENT
2
1
0
−1
−2
LIGHTNESS
1
0
2
−2
2.0
2.0
2.0
2.0
10
5
0
−5
−10
ZA OFFSET, ARCMIN
−10
−5
0
5
10
AZ OFFSET, GREAT−CIRCLE ARCMIN


STOKES U, PERCENT
2
1
0
−1
−2
LIGHTNESS
1
0
2
I
−2
2.0
2.0
2.0
2.0
ZA OFFSET, ARCMIN
10
5
0
−5
−10
AZ OFFSET, GREAT−CIRCLE ARCMIN
−10
−5
0
5
10


STOKES V, PERCENT
1.50
0.75
0.00
−0.75
−1.50
LIGHTNESS
1
0
1.5
I
−1.5
1.0
1.0
10
5
0
−5
−10
ZA OFFSET, ARCMIN
−10
−5
0
5
10
AZ OFFSET, GREAT−CIRCLE ARCMIN

THE EFFECT ON ASTRONOMICAL POLARIZATION MEASUREMENTS

Large-scale features have spatial structure of *Stokes I*.

Express this variation by a two-dimensional Taylor expansion. ***Beam squint*** **responds to the *first* derivative;** ***beam squash*** **and** ***beam squoosh*** **respond to the *second*.**

The polarized beam structure interacts with the Stokes I derivatives to produce FAKE RESULTS in the *polarized* Stokes parameters (Q, U, V). The effects are exacerbated by the polarized sidelobes, which are further from beam center.

Stokes I derivatives of 1 K arcmin^{-1} and second derivative 1 K arcmin^{-2} (values which are not necessarily realistic) yield fake results for Stokes $Q, U \sim 0.3$ K. For Stokes V the contributions are about ten times smaller, ~ 0.03 K.

The fractional polarization of extended emission tends to be small, so spatial gradients in I can be very serious. For example, if the central velocity of the 21-cm line has a spatial gradient $\frac{dv}{d\theta} = 1$ km s^{-1} deg^{-1} we get $B_{fake} \sim 1.1\ \mu$G.

Correcting for these effects at Arecibo is a complicated business because of the PA variation with azimuth and zenith angle. It is also an uncertain business, especially for (Q, U) and somewhat less so for V, because these variations are unpredictable and must be determined empirically.

We believe that corrections at the GBT will be much more straightforward because the sidelobes are so small; this work is in progress.